

GLOBAL EXISTENCE AND DECAY ESTIMATES FOR THE HEAT EQUATION WITH POWER-EXPONENTIAL NONLINEARITIES

MOHAMED MAJDOUB AND SLIM TAYACHI

ABSTRACT. In this paper we consider the problem: $\partial_t u - \Delta u = f(u)$, $u(0) = u_0 \in \exp L^p(\mathbb{R}^N)$, where $p > 1$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ having an exponential growth at infinity with $f(0) = 0$. We prove local well-posedness in $\exp L_0^p(\mathbb{R}^N)$ for $f(u) \sim e^{|u|^q}$, $0 < q \leq p$, $|u| \rightarrow \infty$. However, if for some $\lambda > 0$, $\liminf_{s \rightarrow \infty} (f(s) e^{-\lambda s^p}) > 0$, then non-existence occurs in $\exp L^p(\mathbb{R}^N)$. Under smallness condition on the initial data and for exponential nonlinearity f such that $|f(u)| \sim |u|^m$ as $u \rightarrow 0$, $m \geq 1 + \frac{2p}{N}$, we show that the solution is global. In particular, $p - 1 > 0$ sufficiently small is allowed. Moreover, we obtain decay estimates in Lebesgue spaces for large time which depend on m .

1. INTRODUCTION

In this paper we study the Cauchy problem:

$$\begin{cases} \partial_t u - \Delta u = f(u), \\ u(0) = u_0 \in \exp L^p(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $p > 1$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ having an exponential growth at infinity with $f(0) = 0$.

As is a standard practice, we study (1.1) via the associated integral equation:

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(u(s)) ds, \quad (1.2)$$

where $e^{t\Delta}$ is the linear heat semi-group.

For exponential nonlinearities $f(u) \sim e^{|u|^p}$, u large, the better space is the so-called Orlicz space $\exp L^p(\mathbb{R}^N)$, $p > 1$ which is a generalization of Lebesgue spaces and contains $L^q(\mathbb{R}^N)$ for every $p \leq q < \infty$. See [4, 5, 6, 10, 8] for the case $p = 2$. The Orlicz space $\exp L^p(\mathbb{R}^N)$ is defined as follows

$$\exp L^p(\mathbb{R}^N) = \left\{ u \in L_{loc}^1(\mathbb{R}^N); \int_{\mathbb{R}^N} \left(e^{\frac{|u(x)|^p}{\lambda^p}} - 1 \right) dx < \infty, \text{ for some } \lambda > 0 \right\},$$

endowed with the Luxembourg norm

$$\|u\|_{\exp L^p(\mathbb{R}^N)} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left(e^{\frac{|u(x)|^p}{\lambda^p}} - 1 \right) dx \leq 1 \right\}.$$

Since the space of smooth compactly supported functions $C_0^\infty(\mathbb{R}^N)$ is not dense in the Orlicz space $\exp L^p(\mathbb{R}^N)$ (see [6, 5]), we use the space $\exp L_0^p(\mathbb{R}^N)$ which is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the Luxemburg norm $\|\cdot\|_{\exp L^p(\mathbb{R}^N)}$. It is known that [6]

$$\exp L_0^p(\mathbb{R}^N) = \left\{ u \in L_{loc}^1(\mathbb{R}^N); \int_{\mathbb{R}^N} \left(e^{\alpha |u(x)|^p} - 1 \right) dx < \infty, \text{ for every } \alpha > 0 \right\}. \quad (1.3)$$

It is easy to show that the linear heat semi-group $e^{t\Delta}$ is continuous at $t = 0$ in $\exp L_0^p(\mathbb{R}^N)$. However, this is not the case in $\exp L^p(\mathbb{R}^N)$.

In the sequel, we adopt the following definitions of weak, weak-mild and classical solutions to Cauchy problem (1.1).

Definition 1.1 (Weak solution). *Let $u_0 \in \exp L_0^p(\mathbb{R}^N)$ and $T > 0$. We say that the function $u \in C([0, T]; \exp L_0^p(\mathbb{R}^N))$ is a weak solution of (1.1) if u verifies (1.1) in the sense of distribution and $u(t) \rightarrow u_0$ in the weak* topology as $t \searrow 0$.*

Definition 1.2 (Weak-mild solution). *We say that $u \in L^\infty(0, T; \exp L^p(\mathbb{R}^N))$ is a weak-mild solution of the Cauchy problem (1.1) if u satisfies the associated integral equation (1.2) in $\exp L^p(\mathbb{R}^N)$ for almost all $t \in (0, T)$ and $u(t) \rightarrow u_0$ in the weak* topology as $t \searrow 0$.*

Definition 1.3 ($\exp L^p$ -classical solution). *Let $u_0 \in \exp L^p(\mathbb{R}^N)$ and $T > 0$. A function $u \in C((0, T]; \exp L^p(\mathbb{R}^N)) \cap L_{loc}^\infty(0, T; L^\infty(\mathbb{R}^N))$ is said to be $\exp L^p$ -classical solution of (1.1) if $u \in C^{1,2}((0, T) \times \mathbb{R}^N)$, verifies (1.1) in the classical sense and $u(t) \rightarrow u_0$ in the weak* topology as $t \searrow 0$.*

We are first interested in the local well-posedness. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $\exp L_0^p(\mathbb{R}^N)$, we are able to prove local existence and uniqueness to (1.1) for initial data in $\exp L_0^p(\mathbb{R}^N)$. We assume that the nonlinearity f satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \leq C|u - v|(e^{\lambda|u|^p} + e^{\lambda|v|^p}), \quad \forall u, v \in \mathbb{R}, \quad (1.4)$$

for some constants $C > 0$, $p > 1$ and $\lambda > 0$. Our first main result reads as follows.

Theorem 1.4 (Local well-posedness). *Suppose that f satisfies (1.4). Given any $u_0 \in \exp L_0^p(\mathbb{R}^N)$ with $p > 1$, there exist a time $T = T(u_0) > 0$ and a unique weak solution $u \in C([0, T]; \exp L_0^p(\mathbb{R}^N))$ to (1.1).*

We stress that the density of $C_0^\infty(\mathbb{R}^N)$ in $\exp L_0^p(\mathbb{R}^N)$ is crucial in the above Theorem. In fact we have obtained the following non-existence result in $\exp L^p(\mathbb{R}^N)$.

Theorem 1.5 (Non-existence). *Let $p > 1$, $\alpha > 0$ and*

$$\Phi_\alpha(x) = \begin{cases} \alpha \left(-\log |x| \right)^{\frac{1}{p}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \quad (1.5)$$

Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, positive on $[0, \infty)$ and satisfies

$$\liminf_{s \rightarrow \infty} \left(f(s) e^{-\lambda s^p} \right) > 0, \quad \lambda > 0. \quad (1.6)$$

Then $\Phi_\alpha \in \exp L^p(\mathbb{R}^N) \setminus \exp L_0^p(\mathbb{R}^N)$ and there exists $\alpha_0 > 0$ such that for every $\alpha \geq \alpha_0$ and $T > 0$ the Cauchy problem (1.1) with $u_0 = \Phi_\alpha$ has no nonnegative $\exp L^p$ -classical solution in $[0, T]$.

The results of Theorems 1.4-1.5 are known for $p = 2$ in [6].

Our next interest is the global existence and the decay estimate. It depends on the behavior of the nonlinearity $f(u)$ near $u = 0$. The following behavior near 0 will be allowed

$$|f(u)| \sim |u|^m,$$

where $\frac{N(m-1)}{2} \geq p$. More precisely, we suppose that the nonlinearity f satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \leq C |u - v| \left(|u|^{m-1} e^{\lambda|u|^p} + |v|^{m-1} e^{\lambda|v|^p} \right), \quad \forall u, v \in \mathbb{R}, \quad (1.7)$$

where $\frac{N(m-1)}{2} \geq p \geq 1$, $C > 0$, and $\lambda > 0$ are constants. Our aim is to obtain global existence to the Cauchy problem (1.1) for small initial data in $\exp L^p(\mathbb{R}^N)$. We have obtained the following.

Theorem 1.6 (Global existence). *Let $m \geq 1 + \frac{2p}{N}$, $p > 1$. Assume that the nonlinearity f satisfies (1.7). Then, there exists a positive constant $\varepsilon > 0$ such that every initial data $u_0 \in \exp L^p(\mathbb{R}^N)$ with $\|u_0\|_{\exp L^p(\mathbb{R}^N)} \leq \varepsilon$, there exists a weak-mild solution $u \in L^\infty(0, \infty; \exp L^p(\mathbb{R}^N))$ of the Cauchy problem (1.1) satisfying*

$$\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{\exp L^p(\mathbb{R}^N)} = 0. \quad (1.8)$$

Moreover, there exists a constant $C > 0$ such that,

$$\|u(t)\|_r \leq C t^{-\sigma}, \quad \forall t > 0, \quad (1.9)$$

where

$$\sigma = \frac{1}{m-1} - \frac{N}{2r} > 0,$$

with r sufficiently large and we may take $r = \infty$ if $N < \frac{2p}{p-1}$.

Hereafter, $\|\cdot\|_r$ denotes the norm in the Lebesgue space $L^r(\mathbb{R}^N)$, $1 \leq r \leq \infty$. We mention that the assumption for the nonlinearity covers the cases

$$f(u) = \pm |u|^{m-1} u e^{|u|^p}, \quad m \geq 1 + \frac{2p}{N}.$$

The global existence part of Theorem 1.6 is known for $p = 2$ (see [5]). The estimate (1.9) was obtained in [5] for $p = 2$ and $m = 1 + \frac{4}{N}$. This is improved in [8] for $p = 2$ and any $m \geq 1 + \frac{4}{N}$. The fact that estimate (1.9) depends on the smallest power of the nonlinearity $f(u)$ is known in [11] but only for nonlinearities having polynomial growth.

Using similar arguments as in [14], we can show the following lower estimate of the blow-up rate.

Theorem 1.7 (Blow-up rate). *Assume that the nonlinearity f satisfies (4.1). Let $u_0 \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in C([0, T_{\max}); L^p \cap L^\infty)$ be the maximal solution of (1.1). If $T_{\max} < \infty$, then we have*

$$\|u(t)\|_{L^p \cap L^\infty} \geq C \left(-\log(T_{\max} - t) \right)^{1/p}, \quad 0 \leq t < T_{\max}.$$

The rest of this paper is organized as follows. In the next section, we collect some basic facts and useful tools about Orlicz spaces. Section 3 is devoted to some crucial estimates on the linear heat semi-group. The sketches of the proofs of Theorems 1.4-1.5-1.6 are done in Section 4. In all this paper, C will be a positive constant which may have different values at different places. Also, $L^r(\mathbb{R}^N)$, $\exp L^r(\mathbb{R}^N)$, $\exp L_0^r(\mathbb{R}^N)$ will be written respectively L^r , $\exp L^r$ and $\exp L_0^r$.

2. ORLICZ SPACES: BASIC FACTS AND USEFUL TOOLS

Let us recall the definition of the so-called Orlicz spaces on \mathbb{R}^N and some related basic facts. For a complete presentation and more details, we refer the reader to [1, 9, 12].

Definition 2.1.

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \rightarrow 0^+} \phi(s), \quad \lim_{s \rightarrow \infty} \phi(s) = \infty.$$

We say that a function $u \in L_{loc}^1(\mathbb{R}^N)$ belongs to $L^\phi(\mathbb{R}^N)$ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.$$

We denote then

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}. \quad (2.1)$$

It is known that $(L^\phi(\mathbb{R}^N), \|\cdot\|_{L^\phi})$ is a Banach space. Note that, if $\phi(s) = s^p$, $1 \leq p < \infty$, then L^ϕ is nothing else than the Lebesgue space L^p . Moreover, for $u \in L^\phi$ with $K := \|u\|_{L^\phi} > 0$, we have

$$\left\{ \lambda > 0, \quad \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\} = [K, \infty[.$$

In particular

$$\int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\|u\|_{L^\phi}}\right) dx \leq 1. \quad (2.2)$$

We also recall the following well known properties.

Proposition 2.2. *We have*

- (i) $L^1 \cap L^\infty \subset L^\phi(\mathbb{R}^N) \subset L^1 + L^\infty$.
- (ii) *Lower semi-continuity:*

$$u_n \rightarrow u \quad a.e. \quad \implies \quad \|u\|_{L^\phi} \leq \liminf \|u_n\|_{L^\phi}.$$

- (iii) *Monotonicity:*

$$|u| \leq |v| \quad a.e. \quad \implies \quad \|u\|_{L^\phi} \leq \|v\|_{L^\phi}.$$

- (iv) *Strong Fatou property:*

$$0 \leq u_n \nearrow u \quad a.e. \quad \implies \quad \|u_n\|_{L^\phi} \nearrow \|u\|_{L^\phi}.$$

Denote by

$$L_0^\phi(\mathbb{R}^N) = \left\{ u \in L_{loc}^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty, \quad \forall \lambda > 0 \right\}.$$

It can be shown (see for example [6]) that

$$L_0^\phi(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{L^\phi} = \text{the closure of } C_0^\infty(\mathbb{R}^N) \text{ in } L^\phi(\mathbb{R}^N).$$

Clearly $L_0^\phi(\mathbb{R}^N) = L^\phi(\mathbb{R}^N)$ for $\phi(s) = s^p$, $p \geq 1$, but this is not the case for any ϕ (see [6]).

When $\phi(s) = e^{s^r} - 1$, we denote the space $L^\phi(\mathbb{R}^N)$ by $\exp L^r$ and $L_0^\phi(\mathbb{R}^N)$ by $\exp L_0^r$.

We have the embedding: $\exp L^r \hookrightarrow L^p$ for every $1 < r \leq p$. More precisely:

Lemma 2.3. *For every $1 < r \leq \rho < \infty$, we have*

$$\|u\|_\rho \leq \left(\Gamma \left(\frac{\rho}{r} + 1 \right) \right)^{\frac{1}{\rho}} \|u\|_{\exp L^r}, \quad (2.3)$$

where $\Gamma(x) := \int_0^\infty \tau^{x-1} e^{-\tau} d\tau$, $x > 0$.

The proof of the previous lemma is similar to that in [10].

We recall that the following properties of the functions Γ and \mathcal{B} given by

$$\mathcal{B}(x, y) = \int_0^1 \tau^{1-x} (1-\tau)^{1-y} d\tau, \quad x, y > 0.$$

We have

$$\mathcal{B}(x, y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}, \quad \forall x, y > 0, \quad (2.4)$$

$$\Gamma(x) \geq C > 0, \quad \forall x > 0, \quad (2.5)$$

$$\Gamma(x+1) \sim \left(\frac{x}{e} \right)^x \sqrt{2\pi x}, \quad \text{as } x \rightarrow \infty, \quad (2.6)$$

and

$$\Gamma(x+1) \leq Cx^{x+\frac{1}{2}}, \quad \forall x \geq 1. \quad (2.7)$$

The following Lemma will be useful in the proof of the global existence.

Lemma 2.4. *Let $\lambda > 0$, $1 \leq p$, $\rho < \infty$ and $M > 0$ such that $\lambda \rho M^p \leq 1$. Assume that*

$$\|u\|_{\exp L^p} \leq M.$$

Then

$$\|e^{\lambda|u|^p} - 1\|_\rho \leq (\lambda \rho M^p)^{\frac{1}{\rho}}.$$

Proof. Write

$$\begin{aligned} \int_{\mathbb{R}^N} \left(e^{\lambda|u|^p} - 1 \right)^\rho dx &\leq \int_{\mathbb{R}^N} \left(e^{\lambda \rho |u|^p} - 1 \right) dx \\ &\leq \int_{\mathbb{R}^N} \left(e^{\lambda \rho M^p \frac{|u|^p}{\|u\|_{\exp L^p}^p}} - 1 \right) dx \\ &\leq \lambda \rho M^p \int_{\mathbb{R}^N} \left(e^{\frac{|u|^p}{\|u\|_{\exp L^p}^p}} - 1 \right) dx \leq \lambda \rho M^p, \end{aligned}$$

where we have used the fact that $e^{\theta s} - 1 \leq \theta(e^s - 1)$, $0 \leq \theta \leq 1$, $s \geq 0$ and (2.2). \square

We state the following proposition which is needed for the local well-posedness in the space $\exp L_0^p$.

Proposition 2.5. *Let $u \in C([0, T]; \exp L_0^p)$. Assume that f satisfies (1.4). Then for every $p \leq r < \infty$ there holds*

$$f(u) \in C([0, T]; L^r).$$

The proof is similar to that given in [8]. Note that we have the embedding $L^p \cap L^\infty \hookrightarrow \exp L_0^p$, for all $p \geq 1$, and

$$\|u\|_{\exp L^p} \leq \frac{1}{(\log 2)^{\frac{1}{p}}} \left(\|u\|_p + \|u\|_\infty \right). \quad (2.8)$$

3. LINEAR ESTIMATES

In this section we establish some results needed for the proofs of the main theorems. We first recall some basic estimates for the linear heat semigroup $e^{t\Delta}$. The solution of the linear heat equation

$$\begin{cases} \partial_t u = \Delta u, & t > 0, x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), \end{cases}$$

can be written as a convolution:

$$u(t, x) = (G_t \star u_0)(x) := (e^{t\Delta} u_0)(x),$$

where

$$G_t(x) := G(t, x) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{N}{2}}}, \quad t > 0, x \in \mathbb{R}^N,$$

is the heat kernel. We will frequently use the $L^r - L^p$ estimate as stated in the Proposition below.

Proposition 3.1. *For all $1 \leq r \leq \rho \leq \infty$, we have*

$$\|e^{t\Delta} \varphi\|_\rho \leq t^{-\frac{N}{2}(\frac{1}{r} - \frac{1}{\rho})} \|\varphi\|_r, \quad \forall t > 0, \forall \varphi \in L^r. \quad (3.1)$$

The following Proposition is a generalization of [5, Lemma 2.2, p 1176].

Proposition 3.2. *Let $1 \leq q \leq p$, $1 \leq r \leq \infty$. Then the following estimates hold:*

- (i) $\|e^{t\Delta} \varphi\|_{\exp L^p} \leq \|\varphi\|_{\exp L^p}, \quad \forall t > 0, \forall \varphi \in \exp L^p.$
- (ii) $\|e^{t\Delta} \varphi\|_{\exp L^p} \leq t^{-\frac{N}{2q}} \left(\log(t^{-\frac{N}{2}} + 1) \right)^{-\frac{1}{p}} \|\varphi\|_q, \quad \forall t > 0, \forall \varphi \in L^q.$
- (iii) $\|e^{t\Delta} \varphi\|_{\exp L^p} \leq \frac{1}{(\log 2)^{\frac{1}{p}}} \left[t^{-\frac{N}{2r}} \|\varphi\|_r + \|\varphi\|_p \right], \quad \forall t > 0, \forall \varphi \in L^r \cap L^p.$

The proof can be done as in [5, 8]. As a consequence we have the following, the proof of which can be done as in [8].

Corollary 3.3. *Let $p > 1$, $N > \frac{2p}{p-1}$, $r > \frac{N}{2}$. Then, for every $g \in L^1 \cap L^r$, we have*

$$\|e^{t\Delta} g\|_{\exp L^p} \leq \kappa(t) \|g\|_{L^1 \cap L^r}, \quad \forall t > 0,$$

where $\kappa \in L^1(0, \infty)$ is given by

$$\kappa(t) = \frac{2}{(\log 2)^{\frac{1}{p}}} \min \left\{ t^{-\frac{N}{2r}} + 1, t^{-\frac{N}{2}} \left(\log(t^{-\frac{N}{2}} + 1) \right)^{-\frac{1}{p}} \right\}.$$

Here we use $\|g\|_{L^1 \cap L^q} = \|g\|_1 + \|g\|_q$.

We will also need the following result for the proofs.

Proposition 3.4. *If $u_0 \in \exp L_0^p$ then $e^{t\Delta} u_0 \in C([0, \infty); \exp L_0^p)$.*

It is known that $e^{t\Delta}$ is a C^0 -semigroup on L^p . By Proposition 3.4, it is also a C^0 -semigroup on $\exp L_0^p$. This is not the case on $\exp L^p$. We have the following result.

Proposition 3.5. *There exist $u_0 \in \exp L^p$ and a constant $C > 0$ such that*

$$\|e^{t\Delta} u_0 - u_0\|_{\exp L^p} \geq C, \quad \forall t > 0. \quad (3.2)$$

The proof of the previous proposition uses the notion of rearrangement of functions and can be done as in [8].

4. SKETCHES OF PROOFS

4.1. Local well-posedness. In this section we prove the existence and the uniqueness of solution to (1.1) in $C([0, T]; \exp L_0^p)$ for some $T > 0$, namely Theorem 1.4. Throughout this section we assume that the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(0) = 0$ and

$$|f(u) - f(v)| \leq C|u - v| \left(e^{\lambda|u|^p} + e^{\lambda|v|^p} \right), \quad \forall u, v \in \mathbb{R} \quad (4.1)$$

for some constants $C > 0$, $\lambda > 0$, $p \geq 1$. We emphasize that, thanks to Proposition 2.5, the Cauchy problem (1.1) admits the equivalent integral formulation (1.2). This is formulated as follows.

Proposition 4.1. *Let $T > 0$ and u_0 be in $\exp L_0^p$. If u belongs to $C([0, T]; \exp L_0^p)$, then u is a weak solution of (1.1) if and only if $u(t)$ satisfies the integral equation (1.2) for any $t \in (0, T)$.*

Now we are ready to prove Theorem 1.4. The idea is to split the initial data $u_0 \in \exp L_0^p$ into a small part in $\exp L^p$ and a smooth one. This will be done using the density of $C_0^\infty(\mathbb{R}^N)$ in $\exp L_0^p$. First we solve the initial value problem with smooth initial data to obtain a local and bounded solution v . Then we consider the perturbed equation satisfied by $w := u - v$ and with small initial data. Now we come to the details. For $\varepsilon > 0$ to be chosen later, we write $u_0 = v_0 + w_0$, where $v_0 \in C_0^\infty(\mathbb{R}^N)$ and $\|w_0\|_{\exp L^p} \leq \varepsilon$. Then, we consider the two Cauchy problems:

$$(\mathcal{P}_1) \quad \begin{cases} \partial_t v - \Delta v = f(v), & t > 0, x \in \mathbb{R}^N, \\ v(0) = v_0, \end{cases}$$

and

$$(\mathcal{P}_2) \quad \begin{cases} \partial_t w - \Delta w = f(w + v) - f(v), & t > 0, x \in \mathbb{R}^N, \\ w(0) = w_0. \end{cases}$$

Using similar arguments as in [8], we can prove the following existence result concerning (\mathcal{P}_1) .

Proposition 4.2. *Let $v_0 \in L^p \cap L^\infty$. Then there exist a time $T > 0$ and a solution $v \in C([0, T], \exp L_0^p) \cap L^\infty(0, T; L^\infty)$ to (\mathcal{P}_1) .*

We obtain the following concerning problem (\mathcal{P}_2) .

Proposition 4.3. *Let $T > 0$ and $v \in L^\infty(0, T; L^\infty)$ given by Proposition 4.2. Let $w_0 \in \exp L_0^p$. Then for $\|w_0\|_{\exp L^p} \leq \varepsilon$, with $\varepsilon > 0$ small enough, there exist a time $\tilde{T} = \tilde{T}(w_0, \varepsilon, v) > 0$ and a solution $w \in C([0, \tilde{T}], \exp L_0^p)$ to problem (\mathcal{P}_2) .*

The proof of Proposition 4.3 uses the following lemma.

Lemma 4.4. *Let $v \in L^\infty$ and $w_1, w_2 \in \exp L^p$ with $\|w_1\|_{\exp L^p}, \|w_2\|_{\exp L^p} \leq M$ for some constant $M > 0$. Let $p \leq q < \infty$, and assume that $2^p \lambda q M^p \leq 1$ where λ is given by (4.1). Then there exists a constant $C > 0$ such that*

$$\left\| f(w_1 + v) - f(w_2 + v) \right\|_q \leq C e^{2^{p-1} \lambda \|v\|_\infty^p} \left\| w_1 - w_2 \right\|_{\exp L^p}.$$

Proof of the Lemma 4.4. By the assumption (4.1) on f , we have

$$\begin{aligned}
\left\| f(w_1 + v) - f(w_2 + v) \right\|_q &\leq C \left\| |w_1 - w_2| \left(e^{2^{p-1}\lambda|w_1|^p + 2^{p-1}\lambda|v|^p} + e^{2^{p-1}\lambda|w_2|^p + 2^{p-1}\lambda|v|^p} \right) \right\|_q \\
&\leq e^{2^{p-1}\lambda\|v\|_\infty^p} \left(2C \left\| w_1 - w_2 \right\|_q + C \left\| |w_1 - w_2| \left(e^{2^{p-1}\lambda|w_1|^p} - 1 \right) \right\|_q \right) \\
&\quad + C e^{2^{p-1}\lambda\|v\|_\infty^p} \left\| |w_1 - w_2| \left(e^{2^{p-1}\lambda|w_2|^p} - 1 \right) \right\|_q \\
&\leq e^{2^{p-1}\lambda\|v\|_\infty^p} \left(2C \left\| w_1 - w_2 \right\|_q + C \left\| w_1 - w_2 \right\|_{2q} \left\| e^{2^{p-1}\lambda|w_1|^p} - 1 \right\|_{2q} \right) \\
&\quad + C e^{2^{p-1}\lambda\|v\|_\infty^p} \left\| w_1 - w_2 \right\|_{2q} \left\| e^{2^{p-1}\lambda|w_2|^p} - 1 \right\|_{2q} \\
&\leq C e^{2^{p-1}\lambda\|v\|_\infty^p} \left\| w_1 - w_2 \right\|_{\exp L^p},
\end{aligned}$$

where we have used Hölder inequality, Lemma 2.3, Lemma 2.4 and the fact that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, for every $a, b \geq 0$ and any $p \geq 1$. This finishes the proof of Lemma 4.4. \square

4.2. Non-existence. The following lemma is the key of the proof of Theorem 1.5.

Lemma 4.5. *Let $p > 1$, $\alpha > 0$. Let Φ_α be given by (1.5) and f , $\lambda > 0$ be as in (1.6). Then, there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha_0$, $\varepsilon > 0$ and $r > 0$, we have*

$$\int_0^\varepsilon \int_{|x| < r} \exp \left(\lambda \left(e^{t\Delta} \Phi_\alpha \right)^p \right) dx dt = \infty.$$

Proof of Lemma 4.5. Let $B(a, \rho)$ denotes the open ball centered at $a \in \mathbb{R}^N$ and with radius $\rho > 0$. Fix $\varepsilon, r > 0$. For $\rho = \min(r, \frac{1}{4})$, we have $B(3x, |x|) \subset B(0, 1)$ for any $|x| < \rho$. Therefore, for any $|x| < \rho$, it holds

$$\begin{aligned}
\left(e^{t\Delta} \Phi_\alpha \right)(x) &= \frac{1}{(4\pi t)^{N/2}} \int_{|x| < 1} e^{-\frac{|x-y|^2}{4t}} \Phi_\alpha(y) dy \\
&\geq \frac{\alpha}{(4\pi t)^{N/2}} \int_{|y-3x| < |x|} e^{-\frac{|x-y|^2}{4t}} \left(-\log |y| \right)^{\frac{1}{p}} dy \\
&\geq C\alpha \left(\frac{|x|^2}{t} \right)^{N/2} e^{-\frac{9}{4}\frac{|x|^2}{t}} \left(-\log 4|x| \right)^{1/p}.
\end{aligned}$$

Let $\eta = \min(\varepsilon, \rho^2)$. Then, for any $0 < t < \eta$, we have $B(0, \sqrt{t}) \subset B(0, \rho)$. Hence

$$\begin{aligned}
\int_0^\varepsilon \int_{|x| < r} \exp \left(\lambda \left(e^{t\Delta} \Phi_\alpha \right)^p \right) dx dt &\geq \int_0^\eta \int_{|x| < \rho} \exp \left(\lambda \left(e^{t\Delta} \Phi_\alpha \right)^p \right) dx dt \\
&\geq \int_0^\eta \int_{\frac{\sqrt{t}}{2} < |x| < \sqrt{t}} \exp(-C\lambda\alpha^p \log(4|x|)) dx dt \\
&\geq C_\alpha \int_0^\eta t^{\frac{N}{2} - \frac{C\lambda\alpha^p}{2}} dt = \infty,
\end{aligned}$$

for $\alpha \geq \alpha_0 := \left(\frac{N+2}{C\lambda} \right)^{1/p}$. This finishes the proof of Lemma 4.5. \square

The proof of Theorem 1.5 follows similar arguments as in [6] and uses the previous Lemma.

4.3. Global Existence. This section is devoted to the proof of Theorem 1.6. The proof uses a fixed point argument on the associated integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (f(u))(s) ds, \quad (4.2)$$

where $\|u_0\|_{\exp L^p} \leq \varepsilon$, with small $\varepsilon > 0$ to be fixed later. The nonlinearity f satisfies $f(0) = 0$ and

$$|f(u) - f(v)| \leq C |u - v| \left(|u|^{m-1} e^{\lambda|u|^p} + |v|^{m-1} e^{\lambda|v|^p} \right), \quad (4.3)$$

for some constants $C > 0$ and $\lambda > 0$, $p \geq 1$ and m is larger than $1 + \frac{2p}{N}$. We will perform a fixed point argument on a suitable metric space. For $M > 0$ we introduce the space

$$Y_M := \left\{ u \in L^\infty(0, \infty, \exp L^p); \sup_{t>0} t^\sigma \|u(t)\|_r + \|u\|_{L^\infty(0, \infty; \exp L^p)} \leq M \right\},$$

where $r > \frac{N(m-1)}{2} \geq p$ and

$$\sigma = \frac{1}{m-1} - \frac{N}{2r} = \frac{N}{2} \left(\frac{2}{N(m-1)} - \frac{1}{r} \right) > 0.$$

Endowed with the metric $d(u, v) = \sup_{t>0} (t^\sigma \|u(t) - v(t)\|_r)$, Y_M is a complete metric space. This follows by Proposition 2.2. For $u \in Y_M$, we define $\Phi(u)$ by

$$\Phi(u)(t) := e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (f(u(s))) ds. \quad (4.4)$$

By Proposition 3.2 (i), Proposition 3.1 and Lemma 2.3, we have

$$\|e^{t\Delta} u_0\|_{\exp L^p} \leq \|u_0\|_{\exp L^p},$$

and

$$\begin{aligned} t^\sigma \|e^{t\Delta} u_0\|_r &\leq t^\sigma t^{-\frac{N}{2} \left(\frac{2}{N(m-1)} - \frac{1}{r} \right)} \|u_0\|_{\frac{N(m-1)}{2}} \\ &= \|u_0\|_{\frac{N(m-1)}{2}} \leq C \|u_0\|_{\exp L^p}, \end{aligned}$$

where we have used $1 \leq p \leq \frac{N(m-1)}{2} < r$.

To estimate $\|\int_0^t e^{(t-s)\Delta} f(u(s)) ds\|_{\exp L^p}$, we use Proposition 3.2 and Corollary 3.3 as well as Lemma 2.3. We treat separately the cases $N > \frac{2p}{p-1}$, $N = \frac{2p}{p-1}$ and $N < \frac{2p}{p-1}$. The proof is done similarly as in [8]. Hence Φ applies Y_M into it self.

The proof of the fact that Φ is a contraction, uses Lemma 2.3, Proposition 3.2, Proposition 3.1, Properties of the beta and gamma functions and is similar to that of [8]. The conclusion follows then by Banach fixed point Theorem.

We will now prove the statement (1.8). For $q \geq \frac{N}{2}$ and $q \geq p$, we have

$$\begin{aligned} &\|u(t) - e^{t\Delta} u_0\|_{\exp L^p} \\ &\leq \int_0^t \|e^{(t-s)\Delta} f(u(s))\|_{\exp L^p} ds \\ &\leq C \int_0^t \|e^{(t-s)\Delta} f(u(s))\|_p ds + C \int_0^t \|e^{(t-s)\Delta} f(u(s))\|_\infty ds \\ &\leq C \int_0^t \|f(u(s))\|_p ds + C \int_0^t (t-s)^{-\frac{N}{2q}} \|f(u(s))\|_q ds. \end{aligned} \quad (4.5)$$

Now, let us estimate $\|f(u(t))\|_r$ for $r = p, q$. We have

$$|f(u)| \leq C|u|^m e^{\lambda|u|^p}.$$

Therefore, we obtain

$$\|f(u)\|_r \leq C\|u\|^m (e^{\lambda|u|^p} - 1 + 1)\|_r.$$

By using Hölder inequality and Lemma 2.3, we obtain

$$\begin{aligned} \|f(u)\|_r &\leq C\|u\|_{2mr}^m \|e^{\lambda|u|^p} - 1\|_{2r} + \|u\|_{mr}^m \\ &\leq C\|u\|_{\exp L^p}^m \left(\|e^{\lambda|u|^p} - 1\|_{2r} + 1 \right). \end{aligned}$$

Using Lemma 2.4 we conclude that

$$\|f(u)\|_r \leq C\|u\|_{\exp L^p}^m \left((2\lambda r M^p)^{\frac{1}{2r}} + 1 \right) \leq C\|u\|_{\exp L^p}^m. \quad (4.6)$$

Substituting (4.6) in (4.5), we have

$$\begin{aligned} \|u(t) - e^{t\Delta} u_0\|_{\exp L^p} &\leq C \int_0^t \left[\|u\|_{\exp L^p}^m + (t-s)^{-\frac{N}{2q}} \|u\|_{\exp L^p}^m \right] ds \\ &\leq Ct\|u\|_{L^\infty(0,\infty;\exp L^p)}^m + Ct^{1-\frac{N}{2q}} \|u\|_{L^\infty(0,\infty;\exp L^p)}^m \\ &\leq C_1 t + C_2 t^{1-\frac{N}{2q}}, \end{aligned}$$

where C_1, C_2 are finite positive constants. This gives

$$\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{\exp L^p} = 0,$$

and proves statement (1.8).

Finally the fact that $u(t) \rightarrow u_0$ as $t \rightarrow 0$ in the weak* topology can be done as in [5]. So we omit the proof here.

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UNIVERSITÉ DE TUNIS EL MANAR, FACULTÉ DES SCIENCES DE TUNIS, DÉPARTEMENT DE MATHÉMATIQUES,
LABORATOIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES (LR03ES04), 2092 TUNIS, TUNISIE
E-mail address: mohamed.majdoub@fst.rnu.tn

UNIVERSITÉ DE TUNIS EL MANAR, FACULTÉ DES SCIENCES DE TUNIS, DÉPARTEMENT DE MATHÉMATIQUES,
LABORATOIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES (LR03ES04), 2092 TUNIS, TUNISIE
E-mail address: slim.tayachi@fst.rnu.tn